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ON CONSERVATIVE FINITE ELEMENT FORMULATIONS OF THE INVISCID BOUSSINESQ EQUATIONS

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SUMMARY

The finite element discretization of the inviscid Boussinesq equations is studied with particular emphasis on the conservation properties of the discrete equations. Methods which conserve the total energy, total temperature and total temperature squared, or two of the above mentioned quantities, are presented. The effect of time discretization, and other numerical errors, on the conservation laws is considered. Finally, the theory is supported and illustrated by several numerical experiments.

KEY WORDS Finite Elements Conservation Forms Inviscid Boussinesq Flow

1. INTRODUCTION

In this paper we consider the inviscid, time-dependent Boussinesq equations; our purpose is to construct finite element approximations which conserve the total energy ($\bar{E}$), total temperature ($\bar{T}$) and total temperature squared ($\bar{T^2}$), or at least two of the above quantities. Though the use of conservative forms for discrete equations is widespread in the finite difference literature, this approach seems to have received little attention from finite element workers. Fix[1] studied a finite element model for isothermal ocean circulation problems using the stream function vorticity formulation. This method conserved kinetic energy, vorticity and mean square vorticity (or enstrophy), and, as a result, he was able to show that the method would not suffer from aliasing errors to any great extent. Lee *et al.*[2] considered the inviscid Boussinesq problem and concluded that it was possible to conserve one of the total energy, total temperature or total temperature squared, but they were unable to find a method which conserved two or indeed all three of the quantities. They found that a method which conserved the quadratic quantities was most stable in terms of time integration, whereas the standard advective form, which does not conserve any of the three quantities mentioned above, performed very poorly in time integrations.

2. CONSERVATION PROPERTIES OF THE CONTINUUM EQUATIONS

The equations for an inviscid, Boussinesq fluid in a two-dimensional region Ω, with boundary Γ are:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p-\rho\mathbf{g}\gamma T, \quad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot\mathbf{u}=0, \quad \text{in } \Omega, \tag{2}$$

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T=0, \quad \text{in } \Omega, \tag{3}$$

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where ρ is the density, $\mathbf{u}$ the velocity, p the pressure, $\mathbf{g}$ the acceleration due to gravity, γ the volumetric coefficient of thermal expansion and T the temperature. We shall consider the case of a contained flow, so that appropriate boundary conditions are

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } \Gamma, \tag{4}$$

where $\mathbf{n}$ is the outward pointing, unit normal vector on Γ.

From (3) we see that

$$\frac{\partial}{\partial t} T^n + \mathbf{u} \cdot \nabla T^n = 0,$$

so that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega T^n &= \int_\Omega \frac{\partial T^n}{\partial t} \\ &= -\int_\Omega \mathbf{u} \cdot \nabla T^n \\ &= -\int_\Omega \nabla \cdot (\mathbf{u} T^n) \\ &= -\int_\Gamma (\mathbf{u} \cdot \mathbf{n}) T^n = 0. \end{aligned} \tag{5}$$

Thus the integral of any power of T is constant (and indeed any function of T).

From (1) we find

$$\rho\left(\frac{\partial}{\partial t} \tfrac{1}{2}\mathbf{u}^2 + \mathbf{u} \cdot \nabla \tfrac{1}{2}\mathbf{u}^2\right) = -\mathbf{u} \cdot \nabla p - \rho \mathbf{u} \cdot \mathbf{g} \gamma T,$$

so

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \tfrac{1}{2}\rho \mathbf{u}^2 &= -\int_\Omega \nabla \cdot [(\tfrac{1}{2}\rho \mathbf{u}^2 + p)\mathbf{u}] - \int_\Omega \rho \mathbf{u} \cdot \mathbf{g} \gamma T \\ &= -\int_\Gamma (\mathbf{u} \cdot \mathbf{n})(\tfrac{1}{2}\rho \mathbf{u}^2 + p) - \int_\Omega \rho \mathbf{u} \cdot \mathbf{g} \gamma T. \end{aligned} \tag{6}$$

Now

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho \mathbf{r} \cdot \mathbf{g} \gamma T &= \int_\Omega \rho \mathbf{r} \cdot \mathbf{g} \gamma \frac{\partial T}{\partial t} \\ &= -\int_\Omega \rho \mathbf{r} \cdot \mathbf{g} \gamma \mathbf{u} \cdot \nabla T \\ &= -\rho \gamma \mathbf{g} \cdot \left\{ \int_\Gamma (\mathbf{u} \cdot \mathbf{n}) \mathbf{r} T - \int_\Omega [\mathbf{r}(\nabla \cdot \mathbf{u}) + \mathbf{u}] T \right\} \\ &= \int_\Omega \rho \gamma \mathbf{g} \cdot \mathbf{u} T, \end{aligned} \tag{7}$$

where $\mathbf{r}$ is the position vector.

Thus we find from (4), (6) and (7) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (\tfrac{1}{2}\rho \mathbf{u}^2 + \rho \gamma \mathbf{g} \cdot \mathbf{r} T) = 0. \tag{8}$$

This equation expresses the fact that the total energy of the fluid is constant.

3. GALERKIN DISCRETIZATION

In this section we consider the numerical discretization of the equations. We use the Galerkin form of the finite element method to discretize in space, and time will appear as a parameter. This leads to a set of coupled, nonlinear, ordinary differential equations for the nodal parameters. We shall not show explicitly the dependence of the functions on time.

The starting point for the finite element method is a weak form of equations (1)–(4). We introduce the following notation: Let $H^1(\Omega)$ be the space of all functions having square integrable first derivatives over Ω, let $L_2(\Omega)$ be the space of functions square integrable over Ω and let $\mathbf{H}^1(\Omega) = H^1(\Omega) \times H^1(\Omega)$, be the space of vector valued functions having square integrable first derivatives over Ω. Finally let $\mathbf{H}_0^1(\Omega)$ be that subspace of $\mathbf{H}^1(\Omega)$ whose elements have vanishing normal component on Γ. A weak form of (1)–(4) may be written:

Find $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $T \in H^1(\Omega)$, $p \in L_2(\Omega)$ such that

$$\int_\Omega \left(\rho \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \alpha \mathbf{u} \nabla \cdot \mathbf{u} + \rho \mathbf{g} \gamma T\right) \cdot \mathbf{w} - \int_\Omega p \nabla \cdot \mathbf{w} = 0, \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega), \tag{9}$$

$$\int_\Omega (\nabla \cdot \mathbf{u}) q = 0, \quad \forall q \in L_2(\Omega), \tag{10}$$

$$\int_\Omega \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T + \beta T \nabla \cdot \mathbf{u}\right) R = 0, \quad \forall R \in H^1(\Omega). \tag{11}$$

Provided the solution of (9)–(11) is sufficiently smooth, the weak form is equivalent to the differential form. Note that p is only determined up to a constant. Two parameters, α and β, have been introduced, and from the point of view of the continuum equations, their values make no difference to the solution since $\nabla \cdot \mathbf{u} = 0$. However, when we discretize, $\nabla \cdot \mathbf{u}$ is only zero in an average sense, and different choices of α and β do lead to different discrete forms. Thus α and β provide a certain degree of flexibility in the numerical method. Lee *et al.*[2] used this flexibility in order to obtain conservation properties for the discrete equations.

To obtain a set of semi-discrete equations, we replace the spaces $\mathbf{H}_0^1(\Omega)$, $H^1(\Omega)$ and $L_2(\Omega)$ by finite element subspaces $\mathbf{S}_u^h(\Omega)$, $S_T^h(\Omega)$ and $S_p^h(\Omega)$. Here, h is some measure of the element size, and the spaces have the property that, as $h \to 0$, any function in the corresponding infinite-dimensional space can be approximated, with arbitrary accuracy, by a member of the corresponding finite element space.

The semi-discrete problem may be written:

Find $\mathbf{u}^h \in \mathbf{S}_u^h$, $T^h \in S_T^h$, $p \in S_p^h$ such that

$$\int_\Omega \left(\rho \frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \nabla \mathbf{u}^h + \alpha \mathbf{u}^h \nabla \cdot \mathbf{u}^h + \rho \mathbf{g} \gamma T^h\right) \cdot \mathbf{w}^h - \int_\Omega p^h \nabla \cdot \mathbf{w}^h = 0 \quad \forall \mathbf{w}^h \in \mathbf{S}_u^h \tag{12}$$

$$\int_\Omega (\nabla \cdot \mathbf{u}^h) q^h = 0 \quad \forall q^h \in S_p^h \tag{13}$$

$$\int_\Omega \left(\frac{\partial T^h}{\partial t} + \mathbf{u}^h \cdot \nabla T^h + \beta T^h \nabla \cdot \mathbf{u}^h\right) R^h = 0 \quad \forall R^h \in S_T^h. \tag{14}$$

In practice, a basis is chosen for the finite element spaces with the property that each basis function has a relatively small support. This leads to a set of ordinary differential equations with sparse coefficient matrices.

Since the pressure is only unique up to an additive constant, its value must be specified at some node, and at that node, the continuity equation cannot be applied. However, nothing is lost by this process, since the discrete equations are guaranteed to conserve mass globally because of the boundary conditions on $\mathbf{u}$.

We now examine the conditions under which the semi-discrete equations conserve total energy, temperature and temperature squared. From (14) and since $1 \in S_T^h$, we have

$$\int_\Omega \frac{\partial T^h}{\partial t} + \nabla \cdot (\mathbf{u}^h T^h) + (\beta - 1) T^h \nabla \cdot \mathbf{u}^h = 0$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega T^h = (1-\beta) \int_\Omega T^h \nabla \cdot \mathbf{u}^h. \tag{15}$$

Also from (14) and the fact that $T^h \in S_T^h$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega T^{h2} = (1-2\beta) \int_\Omega T^{h2} \nabla \cdot \mathbf{u}^h. \tag{16}$$

From (12) and the fact that $\mathbf{u}^h \in \mathbf{S}_u^h$, we have

$$\int_\Omega \tfrac{1}{2}\rho \frac{\partial}{\partial t} \mathbf{u}^{h2} + \rho \mathbf{u}^h \cdot \mathbf{g}\gamma T^h = (\tfrac{1}{2} - \alpha) \int_\Omega \mathbf{u}^{h2} \nabla \cdot \mathbf{u}^h - \int_\Omega p^h \nabla \cdot \mathbf{u}^h.$$

The last term is zero since $p^h \in S_p^h$, and equation (13) is satisfied.
Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \tfrac{1}{2}\rho \mathbf{u}^{h2} + \int_\Omega \rho \mathbf{u}^h \cdot \mathbf{g}\gamma T^h = (\tfrac{1}{2} - \alpha) \int_\Omega \mathbf{u}^{h2} \nabla \cdot \mathbf{u}^h. \tag{17}$$

The first term on the right-hand side of (17) is the rate of change of the discrete kinetic energy. We must relate the second term to the rate of change of potential energy in the discrete problem.

Now

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho \mathbf{r} \cdot \mathbf{g}\gamma T^h = \int_\Omega \rho \mathbf{r} \cdot \mathbf{g}\gamma \frac{\partial T^h}{\partial t} \tag{18}$$

where $\mathbf{r}$ is the position vector. To proceed further we require that $\mathbf{r} \in S_T^h(\Omega) \times S_T^h(\Omega)$. This will be true provided that the mapping of the elements from local co-ordinates to the region Ω is carried out by means of the basis functions in $S_T^h(\Omega)$, i.e. that the elements are sub-parametric with respect to the temperature field. So, if $\mathbf{r} \in S_T^h(\Omega) \times S_T^h(\Omega)$, we have from (18) and (14)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho \mathbf{r} \cdot \mathbf{g}\gamma T^h = -\rho\gamma \mathbf{g} \cdot \int_\Omega \mathbf{r}(\mathbf{u}^h \cdot \nabla T^h + \beta T^h \nabla \cdot \mathbf{u}^h),$$

using the vector identity

$$\mathbf{r}(\mathbf{u}^h \cdot \nabla) T^h = \nabla \cdot (\mathbf{u}^h \mathbf{r} T^h) - \mathbf{u}^h T^h - \mathbf{r}(\nabla \cdot \mathbf{u}) T^h$$

we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho \mathbf{r} \cdot \mathbf{g}\gamma T^h = -\rho\gamma \mathbf{g} \cdot \int_\Gamma (\mathbf{u}^h \cdot \mathbf{n}) \mathbf{r} T^h + \rho\gamma \mathbf{g} \int_\Omega \mathbf{u}^h T^h + (1-\beta)\rho\gamma \mathbf{g} \cdot \int_\Omega \mathbf{r}(\nabla \cdot \mathbf{u}^h) T^h.$$

Thus

$$\int_\Omega \rho\gamma \mathbf{g} \cdot \mathbf{u}^h T^h = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho \mathbf{r} \cdot \mathbf{g}\gamma T^h + (\beta - 1) \int_\Omega \rho\gamma \mathbf{g} \cdot \mathbf{r}(\nabla \cdot \mathbf{u}^h) T^h \tag{19}$$

and from (17), (18) and (19)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \{\tfrac{1}{2}\rho \mathbf{u}^{h2} + \rho \mathbf{r} \cdot \mathbf{g}\gamma T^h\} = (\tfrac{1}{2} - \alpha) \int_\Omega \mathbf{u}^{h2} \nabla \cdot \mathbf{u}^h + (1 - \beta) \int_\Omega \rho\gamma \mathbf{g} \cdot \mathbf{r}(\nabla \cdot \mathbf{u}^h) T^h. \tag{20}$$

Clearly then, for the semi-discrete equations to conserve temperature (resp. temperature squared, resp. total energy), we have the necessary and sufficient condition that the right-hand side of (15) (resp. (16), resp. (20)) be zero.

By suitable choices of α and β it is possible to arrange for any one of $\bar{T}$, $\overline{T^2}$ or $\bar{E}$ to be conserved. Further, with the particular choice $\alpha = \frac{1}{2}$, $\beta = 1$, the temperature and total energy are conserved. This result is at variance with Lee *et al.*[2] who, due to an error in their analysis of the discrete potential energy, obtained equation (20) but with $-\beta$ multiplying the last term on the right-hand side rather than $(1-\beta)$. They found, however, that total energy was not conserved in time integrations of the semi-discrete equations in this case. Since the above analysis proves that the semi-discrete total energy is constant, the lack of conservation in their calculation must arise from either time integration errors, quadrature errors, convergence errors or rounding errors or possibly a combination of these errors. The effect of the above errors on the conservation property can however be made very small and, in our numerical experiments, total energy, in the case $\alpha = \frac{1}{2}$, $\beta = 1$, was conserved virtually to machine accuracy (see the section on numerical experiments).

If we take $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, then $\overline{T^2}$ is conserved. Further $\bar{T}$ will be conserved if

$$\int_\Omega T^h(\nabla \cdot \mathbf{u}^h) = 0 \tag{21}$$

and $\bar{E}$ will be conserved if

$$\mathbf{g} \cdot \int_\Omega \mathbf{r} T^h(\nabla \cdot \mathbf{u}^h) = 0. \tag{22}$$

The crucial observation is that (21) and (22) will be satisfied provided

$$T^h \in S_p^h(\Omega), \quad \forall\, T^h \in S_T^h(\Omega), \tag{23}$$

and

$$\mathbf{g} \cdot \mathbf{r} T^h \in S_p^h(\Omega), \quad \forall\, T^h \in S_T^h(\Omega). \tag{24}$$

This follows directly from equation (13).

Thus the problem is essentially that of constructing suitable finite element spaces for the various fields. It should be noted that we are not free to choose $S_p^h(\Omega)$ independently of $\mathbf{S}_u^h(\Omega)$. One has to be careful not to make $S_p^h(\Omega)$ too big, otherwise spurious pressure modes appear.[3,4]

In most of the finite element solutions of incompressible flow in primitive variables published to date, the pressure field has been approximated either by a piecewise constant or piecewise linear polynomial (in the latter case it is possible either to have continuous or discontinuous pressure approximations). With a piecewise linear continuous pressure field it is possible to satisfy (23) by using piecewise linear polynomials for the temperature field.

Table I. Summary of conservation properties for various combinations of parameters and temperature interpolation

Formulation	α	β	Temperature interpolation	Quantitive conserved
I	$\frac{1}{2}$	$\frac{1}{2}$	4	$\bar{E}, \bar{T}, \overline{T^2}$
II	$\frac{1}{2}$	$\frac{1}{2}$	8	$\bar{T}, \overline{T^2}$
III	$\frac{1}{2}$	1	8	$\bar{E}, \bar{T}$
IV	0	0	8	$\bar{T}$

This combination would produce semi-discrete equations which conserve total temperature and total temperature squared.

However, linear interpolation for the pressure is clearly inadequate if (24) is to be satisfied. In a recent publication[4] Jackson and Cliffe introduced a new element for incompressible flow calculations in which the pressure field is approximated by the eight-noded quadrilateral serendipity element.[5] The velocities are approximated by a nine-noded element, which has eleven parameters: the nine values at the nodes together with two independent derivatives at the central node. It is worth noting that the additional parameters incur very little extra computational cost in the frontal method of solution, since the front width is virtually unchanged. The shape functions for the element are given in the appendix in terms of the local co-ordinates. Jackson and Cliffe[4] have shown that the element does not suffer from spurious pressure modes.

The higher order interpolation of pressure makes it possible to satisfy equation (24). To see this we note that, in terms of local co-ordinates ξ and η, a basis for the pressure approximation on a given element is the set of monomials:

$$\{1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^2\eta, \xi\eta^2\}.$$

If the temperature field is approximated using the standard four-noded element, a basis is the set of monomials:

$$\{1, \xi, \eta, \xi\eta\}.$$

It is thus clear that (24) will be satisfied if $\mathbf{g} \cdot \mathbf{r}$ is a strictly linear function of ξ and η. This means that the elements cannot be fully isoparametric but must, in fact, be parallelograms.

Four different formulations are summarized in Table I, in terms of the parameters α and β, the temperature interpolation and the conserved quantities. The temperature interpolation is restricted to either the four-noded bilinear element or the eight-noded biquadratic serendipity element. Clearly there are many other possible combinations, but we have only investigated the four in Table I so far.

4. THE EFFECT OF TIME DISCRETIZATION AND OTHER ERRORS ON THE CONSERVATION LAWS

In the previous section we established conservation properties for various semi-discrete forms of the inviscid Boussinesq equations. In this section we consider the effect of various numerical errors on the conservation properties.

Firstly we consider the effect of time discretization. We may write the semi-discrete

equations in the form

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}). \tag{25}$$

In order to preserve the conservation properties we clearly require a nondissipative method of integration for (25). The simplest such method is probably the trapezoidal rule (Crank–Nicholson) which leads to

$$\mathbf{y}^{n+1} - \mathbf{y}^n = \frac{\Delta t}{2}(\mathbf{f}(\mathbf{y}^{n+1}) + \mathbf{f}(\mathbf{y}^n)). \tag{26}$$

However, as Lee *et al.*[2] point out, it is easy to see that whilst the discretization (26) will preserve the conservation properties of linear quantities, for quadratic quantities which are conserved in the semi-discrete equations, the conservation properties are lost.

The alternative second-order nondissipative method namely, the midpoint rule, does preserve the conservation properties of quadratic quantities, and this leads to

$$\mathbf{y}^{n+1} - \mathbf{y}^n = \Delta t \mathbf{f}(\tfrac{1}{2}(\mathbf{y}^{n+1} + \mathbf{y}^n)). \tag{27}$$

It is a straightforward matter to establish that the conservation properties of formulations I–IV are preserved by (27).

The second potential cause of error is in the solution of the non-linear system of equations (27). In this study we have used a Newton–Raphson method to solve (27) and in this case it is easy to see that, in exact arithmetic, linear conservation properties are preserved at each iteration, whereas for quadratic quantities, the conservation properties are, in general, only preserved in the limit of convergence. In practice, of course, the convergence error for a Newton method will be less than machine precision after a few iterations.

Thirdly, since the conservation properties depend on the exact form of the discrete equations, quadrature errors (i.e. errors in evaluating the integrals involving the basis functions) should be avoided. In our program we have used integration schemes which are sufficiently accurate to evaluate all integrals exactly. We note here that allowing general isoparametric transformations of the elements leads to integrals which cannot be evaluated without quadrature error by Gaussian integration schemes.

Finally, all the conservation properties are affected by rounding errors, which, although they cannot be avoided, are usually very small.

5. NUMERICAL EXPERIMENTS

In this section we describe the numerical experiments we have done to support and illustrate the theory presented in the previous sections. The problem we have applied formulations I–IV to is the inviscid flow within a closed cavity, studied by Lee *et al.*[2]

The region and mesh, consisting of twelve, nine-noded rectangular elements, are shown in Figure 1. The initial conditions are

$$\mathbf{u}_0 = \mathbf{0},$$

$$T_0 = \begin{cases} -(1-10x)(1-10y) & 0 \leq x, y \leq 0.1 \\ 0 & \text{otherwise} \end{cases}$$

The boundary conditions for velocity are

$$\mathbf{u} \cdot \mathbf{n} = 0$$

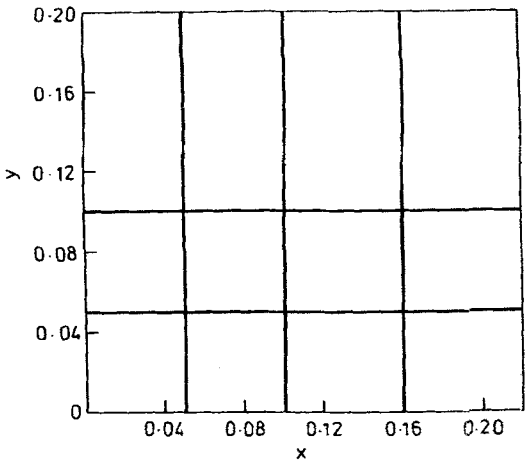

Figure 1. The mesh of twelve, nine-noded, rectangular elements

on Γ (no boundary conditions are required for temperature). The other values used in the computations are $|\mathbf{g}| = 1{\cdot}0(\mathbf{g} \cdot \mathbf{r} = -|\mathbf{g}|\, y)$ and $\rho = \gamma = 1{\cdot}0$. For this problem the total energy, total temperature and total temperature squared are constant.

The program we have used for the calculations is based on the TGSL finite element package developed by the Theory of Fluids Group at Harwell. The time integration algorithm uses an explicit predictor to give a good initial guess for the solution of the corrector (27), and also to choose time steps to keep the local truncation error less than some preassigned value.

The nonlinear corrector equations are solved by the Newton–Raphson algorithm, with the

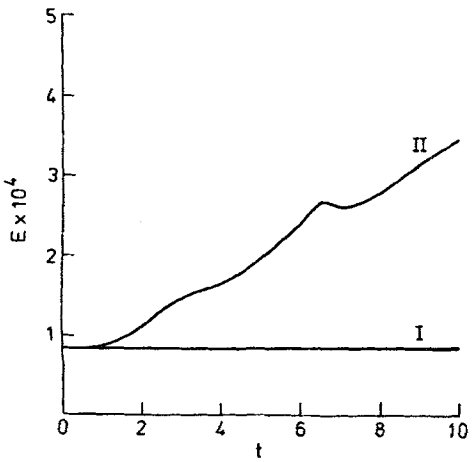

Figure 2. Plot of total energy against time for formulations I and II

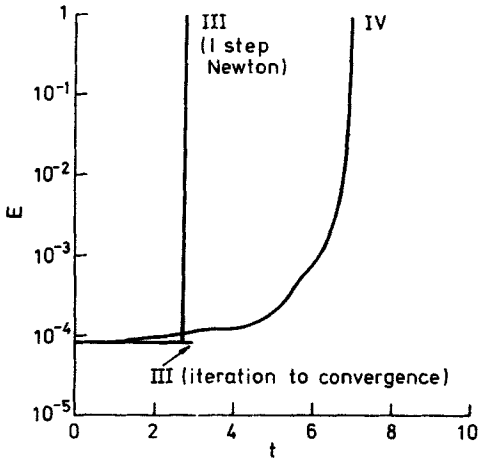

Figure 3. Plot of total energy against time for formulations III and IV

frontal method being used to solve the linear system produced at each iteration. For the calculations presented here, we have either used just one Newton iteration, or else sufficiently many so that the convergence errors were of the order of machine precision. The calculations were run on an IBM 3033 and double precision variables (having approximately 16·8 decimal digits) were used throughout. The integrations were carried out up to time 10 for formulations I and II. For III and IV, the solution 'blew up' at times ~2·7 and ~7 respectively (see Figures 3 and 4). This type of behaviour was reported by Lee *et al.*[2]

All four formulations conserved total temperature very well, the typical relative error

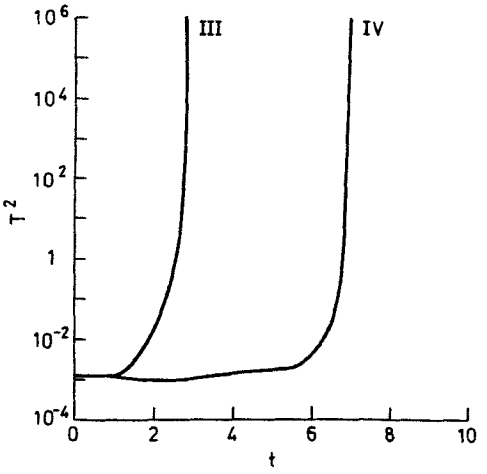

Figure 4. Plot of total temperature squared against time for formulations III and IV

being of order 10^{-14}. Formulation I was run both with a one-step Newton and an iteration-to-convergence method. In the former case, the relative error for total energy and temperature squared was of order 10^{-5}, whereas, in the latter, it was of order 10^{-14}. Thus the one-step Newton method yields perfectly acceptable accuracy. Formulation II conserved total temperature squared very well but the total energy in this case exhibited slow growth as shown in Figure 2. Formulation III suffered from an explosive growth in total temperature squared at time ~2·7 (see Figure 4). The behaviour of the total energy in this case depended on whether a one-step Newton method was used. With the one-step Newton method, the total energy also suffered explosive growth. However, with an iteration-to-convergence method, the total energy was conserved, with a relative error of order 10^{-10}, right up to the time at which the calculation was stopped.

The behaviour of IV was similar to that of III, with the explosive growth coming at a later time.

CONCLUSIONS

We have presented several finite element formulations of the inviscid Boussinesq equations which conserve some or all of the following quantities: total energy, total temperature and total temperature squared. The numerical experiments that we have done are in full agreement with the theoretical properties of the schemes. It is also clear that schemes conserving quadratic quantities are inherently more stable than those conserving only linear quantities.

ACKNOWLEDGEMENTS

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APPENDIX

The basis functions for the element introduced by Jackson and Cliffe are given here in terms of the local co-ordinates (ξ, η). There are eleven basis functions associated with each velocity field. The functions ϕ_i $(i = 1, \ldots, 9)$ are associated with node i and ϕ_{10} and ϕ_{11} are associated with node 9 (see Figure 5).

$$\phi_1 = \tfrac{1}{4}(\xi^2 - \xi)(\eta^2 - \eta)$$
$$\phi_2 = \tfrac{1}{4}(\xi^2 + \xi)(\eta^2 - \eta)$$
$$\phi_3 = \tfrac{1}{4}(\xi^2 + \xi)(\eta^2 + \eta)$$
$$\phi_4 = \tfrac{1}{4}(\xi^2 - \xi)(\eta^2 + \eta)$$
$$\phi_5 = \tfrac{1}{2}(1 - \xi^2)(\eta^2 - \eta^3)$$
$$\phi_6 = \tfrac{1}{2}(\xi^2 + \xi^3)(1 - \eta^2)$$
$$\phi_7 = \tfrac{1}{2}(1 - \xi^2)(\eta^2 + \eta^3)$$
$$\phi_8 = \tfrac{1}{2}(\xi^2 - \xi^3)(1 - \eta^2)$$
$$\phi_9 = (1 - \xi^2)(1 - \eta^2)$$
$$\phi_{10} = (\xi - \xi^3)(1 - \eta^2)$$
$$\phi_{11} = (1 - \xi^2)(\eta - \eta^3)$$

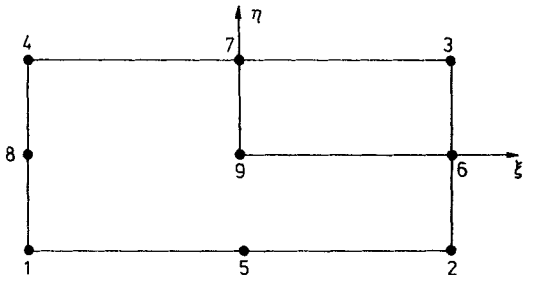

Figure 5. Node numbering and local co-ordinates for the nine-noded rectangular element

The basis functions for the pressure are the same as those for the standard eight-noded biquadratic serendipity element.[5]

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